A Class of Solution for Anisotropic Stars Admitting Conformal Motions in Higher Dimensional Space Time

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Abstract

We provide a new class of interior solutions for anisotropic stars admitting conformal motion. The Einstein's field equations in higher dimensional spherically symmetric charged distribution for specific choices of the density/mass functions are solved. We analyzed the behavior of the model parameters like radial and transverse pressure, density and surface tension.

Keywords : Higher dimension, Matter.

1. Introduction

Anisotropy in fluid pressure could be introduced by the existence of a solid core or by the presence of type 3A superfluid [Kippenhahn and Weigert (1990)], different kinds of phase transitions [Sokolov (1980)], pion condensation [Sawyer (1972)] or by other physical phenomena. On the scale of galaxies, Binney and Tremaine (1987) considered anisotropies in spherical galaxies, from a purely Newtonian point of view. Other source of anisotropy, due to the effects of the slow rotation in a star, has been proposed by Herrera and Santos (1995). The mixture of two gases (e.g., monotonic hydrogen, or ionized hydrogen and electrons) can formally be also described as an anisotropic fluid [Letelier(1980)]. Dev and Gleiser (2002) investigated the effect of anisotropic pressure on the properties of spherically symmetric gravitationally bound objects. Under full general relativistic treatment, exact solution for various form of the equation of state connecting the radial and tangential pressures are obtained. Herrera et al. (1984) studied the consequence of the existence of a one parameter group of conformal motions, for anisotropic matter. They include that for special conformal motions, the stiff equation of state $\rho = p$ is singled out in a unique way, provided the generating conformal vector field is orthogonal to the four velocity. Ruderman (1972) showed that nuclear matter may become anisotropic in the high density region of order 10^{15} gm/cc, which is expected at the core of compact terrestrial objects. Though we lack a complete understanding of the microscopic origin of the pressure anisotropy, the role of pressure anisotropy in the modeling of compact stars is a field of active research.

Rahaman et al. (2010a) provided new class of interior solution for anisotropic stars admitting conformal motions. Harko and Mak (2000) obtained the equation describing the hydrostatic equilibrium of a static anisotropic general relativistic fluid sphere in higher dimensional space time with cosmological constant Λ.
Inspired with the earlier works of, Mak and Harko (2004), Aktas and Yilmaz (2007), Rahaman et al. (2010b), on anisotropic stars admitting a one parameter group of conformal motions in four dimensional general relativity, in this paper we look for a new class of anisotropic star solutions admitting conformal motion in the frame work of higher dimensional general theory of relativity.
It is well known that to find the natural relation between geometry and matter through the Einstein's equations, it is useful to use the inheritance symmetry. The well known inheritance symmetry under conformal killing vectors (CKV) i.e.,

$$\zeta_\xi g_{\mu\vartheta} = \psi g_{\mu\vartheta}, \qquad (1)$$

The quantity on left hand side is the Lie derivative of metric tensor, describing the interior gravitational field of a compact star with respect to the vector field ξ and ψ is the arbitrary function of r. It ψ is a constant then (1) generates homothetic while $\psi = 0$ result in killing vectors. Conformal killing vectors provide a deeper insight into the spacetime geometry.
This paper is generalization of the work obtained earlier by Rahaman et al. (2010a) in higher dimensional space time.
The paper is organized as follows : in section 2, we have provided the basic equations which describe anisotropic star admitting conformal motions in higher dimensional space time. In section 2.1 and 2.2. we solve this field equations for specific choice of energy density ρ of the

form $\rho = \frac{1}{2^{(n+1)}\pi}\left(\frac{a}{r^n} + 3b\right)$ and mass $m = \frac{1}{2(1+ar^n)}\left(br^{(n+1)}\right)$ respectively. Finally some concluding remark have been made in section 3.

2. Anisotropic stars

To generate a new class of solutions making use of this symmetry, we choose static spherically symmetric space time are in the standard form (chosen units are c=1=G)

$$ds^2 = dt^2 - e^{\lambda(r)}dr^2 - \Omega^2 \qquad (2)$$

with $d\Omega^2 = \theta_1^2 + sin^2\theta_1 d\theta_2^2 + sin^2\theta_1 sin^2\theta_2 d\theta_3^2 + \cdots + \prod_{i=1}^{n-1} sin^2\theta_i d\theta_n^2$,

$\chi^{1,2,3,\cdots,(n+1),(n+2)} = r, \theta_1, \theta_2, \cdots, \theta_{(n+1)}, t,$

where $\vartheta(r)$ and $\lambda(r)$ are yet to be determined.
For an anisotropic matter distribution, with the energy momentum tensor given by

$T_{ij} = diag(\rho, -p_r, -p_t, \cdots, ntimes(-p_t)),$

the Einstein's equations for the metric (2) are obtained as

$$8\pi\rho = -e^{-\lambda}\left[\frac{n(n-1)}{2r^2} + \frac{-n\lambda'}{2r}\right] + \frac{n(n-1)}{2r^2}, \qquad (3)$$

$$-8\pi p_r = -e^{-\lambda}\left[\frac{n(n-1)}{2r^2} + \frac{n\vartheta'}{2r}\right] + \frac{n(n-1)}{2r^2}, \qquad (4)$$

$$8\pi p_t = \frac{e^{-\lambda}}{2}\left[\vartheta'' - \frac{\vartheta'\lambda'}{2} + \frac{\vartheta'^2}{2}\frac{(n-1)(\lambda'-\vartheta')}{r} + \frac{(n-1)(n-2)}{r^2}\right] - \frac{(n-1)(n-2)}{2r^2}, \qquad (5)$$

where prime (') denotes differentiation with respect to r. Now the equation

$$\zeta_\xi g_{\mu\vartheta} = \xi_{\mu;\vartheta} + \xi_{\vartheta;\mu} = \psi g_{\mu\vartheta}, \quad \psi = \psi(\chi^\mu), \qquad (6)$$

for the line element given in (2) generates

$$\xi^1\vartheta' = \psi, \qquad (7)$$
$$\xi^{n+2} = C_1 , \qquad (8)$$
$$\xi^1 = \frac{\psi r}{2}, \qquad (9)$$
$$\xi'\lambda' + 2\xi^1_{,1} = \psi, \qquad (10)$$

where C_1 is a constant.
These consequently imply

$$e^\vartheta = C_2^2 r^2 , \qquad (11)$$
$$e^\lambda = \left(\frac{C_3^2}{\psi}\right)^2 , \qquad (12)$$
$$\xi^\mu = \frac{\psi r}{2}\delta_1^\mu + C_1\delta_{n+1}^\mu , \qquad (13)$$

where C_2 and C_3 are constants of integration. Equation (11)-(13), help us to rewrite Eqs. (3)-(5) in the form

$$8\pi\rho = \frac{n(n-1)}{2r^2}\left[1 - \frac{\psi^2}{C_3^2}\right] - \frac{n\psi\psi'}{C_3^2 r}, \qquad (14)$$

$$-8\pi p_r = -\frac{n(n+1)\psi^2}{2r^2 C_3^2} + \frac{n(n-1)}{2r^2}, \qquad (15)$$

$$8\pi p_t = \frac{n\psi\psi'}{C_3^2 r} + \frac{n(n-1)}{2r^2}\frac{\psi^2}{C_3^2} - \frac{(n-1)(n-2)}{2r^2} . \qquad (16)$$

2.1. Given density profile: $\rho = \frac{1}{2^{(n+1)}\pi}\left(\frac{a}{r^n} + 3b\right)$

In Dev and Gleiser (2002, 2004) model, anisotropic star admits two major types of density distribution $\rho = $ constant and $\rho = r^2$. These two can be constructed in one simple form as shown above. Here a and b are constants which generates various configurations of the star. For ex, by choosing a = 3/7 and b = 0, one may obtain a relativistic Fermi gas. Making use of the density profile as prescribed by Dev and Gleiser (2002, 2004), we rewrite Eq. (14) as

$$\frac{1}{2^{(n-2)}}\left(\frac{a}{r^n} + 3b\right) = \frac{n(n-1)}{2r^2}\left[1 - \frac{\psi^2}{C_3^2}\right] - \frac{n\psi\psi'}{C_3^2 r}, \qquad (17)$$

which can be solved easily to yield

$$\psi = \sqrt{\left(1 - \frac{2^{(3-n)}a}{nr^{(n-2)}}\right)C_3^2 - \frac{6br^2 C_3^2}{n(n+1)2^{(n-2)}} + \frac{C}{r^{(n-1)}}} , \qquad (18)$$

where C is an integration constant. Consequently, we obtain an exact analytical solution in the form

$$e^\vartheta = C_3^2 r^2 , \qquad (19)$$

$$e^\lambda = C_3^2\left[\left(1 - \frac{2^{(3-n)}a}{nr^{(n-2)}}\right)C_3^2 - \frac{6br^2 C_3^2}{n(n+1)2^{(n-2)}} + \frac{C}{r^{(n-1)}}\right]^{-1} . \qquad (20)$$

The two pressures are obtained as

$$p_r = \frac{n(n+1)}{16\pi r^2 C_3^2}\left[\left(1 - \frac{2^{(3-n)}a}{nr^{(n-2)}}\right)C_3^2 - \frac{6br^2 C_3^2}{n(n+1)2^{(n-2)}} + \frac{C}{r^{(n-1)}}\right] - \frac{n(n-1)}{16\pi r^2}, \qquad (21)$$

$$p_t = \frac{n(n-1)}{16\pi r^2 C_3^2}\left[\left(1 - \frac{2^{(3-n)}a}{nr^{(n-2)}}\right)C_3^2 - \frac{6br^2 C_3^2}{n(n+1)2^{(n-2)}} + \frac{C}{r^{(n-1)}}\right] + \frac{n}{8\pi r^2}\left[\frac{(2-n)a2^{(2-n)}C_3^2}{nr^n} - \frac{12bC_3^2}{n(n+1)2^{(n-1)}} + \frac{(1-n)C}{2r^{(n+1)}}\right] - \frac{(n+1)(n-2)}{16\pi r^2}. \qquad (22)$$

The measure of pressure anisotropy is given by

$$\Delta \equiv p_t - p_r$$

$$= \frac{(n-1)}{8\pi r^2} - \frac{n}{8\pi r^2 C_3^2}\left[\left(1 - \frac{2^{(3-n)}a}{nr^{(n-2)}}\right)C_3^2 - \frac{6br^2 C_3^2}{n(n+1)} + \frac{C}{r^{(n-1)}}\right]$$

$$+ \frac{n}{8\pi C_3^2}\left[\frac{(2-n)a2^{(2-n)}C_3^2}{nr^n} - \frac{12bC_3^2}{n(n+1)2^{(n-1)}} + \frac{(1-n)C}{2r^{(n+1)}}\right]. \qquad (23)$$

Here, $\frac{\Delta}{r}$ corresponds to a force due to the anisotropic nature of the star. This force will be repulsive if $\frac{\Delta}{r} > 0$

i.e., $p_t > p_r$ and attractive if $\frac{\Delta}{r} < \mathbf{0}$. In the present model, $\frac{\Delta}{r}$ is given by

$$\frac{\Delta}{r} = \frac{(n-1)}{8\pi r^3} - \frac{n}{8\pi r^3 C_3^2}\left[\left(1 - \frac{2^{(3-n)}a}{nr^{(n-2)}}\right)C_3^2 - \frac{6br^2C_3^2}{n(n+1)2^{(n-2)}} + \frac{C}{r^{(n-1)}}\right] + \frac{n}{8\pi rC_3^2}\left[\frac{(2-n)a2^{(2-n)}C_3^2}{nr^n} - \frac{12bC_3^2}{n(n+1)2^{(n-1)}} + \frac{(1-n)C}{2r^{(n+1)}}\right].$$

(24)

At the surface of the star r = R, we impose the condition that the radial pressure vanishes, i.e., p_r (r = R) = 0, which gives

$$\frac{n(n+1)}{16\pi r_0^2 C_3^2}\left[\left(1 - \frac{2^{(3-n)}a}{nr_0^{(n-2)}}\right)C_3^2 - \frac{6br_0^2C_3^2}{n(n+1)2^{(n-2)}} + \frac{C}{r_0^{(n-1)}}\right] - \frac{n(n-1)}{16\pi r_0^2} = 0.$$

(25)

From (25) we can find out the value of R.
After differentiating (22) with respect to r, we get the value of pressure gradient as

$$8\pi\frac{dp_r}{dr} = -\frac{n(n+1)}{r^3 C_3^2}\left[\left(1 - \frac{2^{(3-n)}a}{nr^{(n-2)}}\right)C_3^2 - \frac{C}{r^{(n-1)}}\right] - \frac{n(n+1)}{2r^3 C_3^2}\left[\left(\frac{(2-n)2^{(3-n)}a}{nr^{(n-1)}}\right)C_3^2 + \frac{12brC_3^2}{n(n+1)2^{(n-2)}} + \frac{(n-1)C}{r^n}\right] + \frac{n(n-1)}{r^3}.$$

(26)

The mass function in this case takes the form

$$m(r) = 2^n\pi\int_0^r\rho(r)x^n dx = \frac{1}{2}\left[ar + \frac{3br^{(n+1)}}{n+1}\right].$$

(27)

2.2. Given density profile : $m(r) = \frac{br^{(n+1)}}{2(1+ar^n)}$.

Let us assume a mass function in higher dimension of the form

$$m(r) = \frac{br^{(n+1)}}{2(1+ar^n)},$$

(28)

where a and b are two arbitrary constants.
Such a mass function has been found to be relevant in the studies of compact stars in higher dimension. As the mass m(r) in (n+2)-dimension is defined as

$$m(r) = 2^n\pi\int_0^r\rho(r)x^n dx,$$

(29)

This is equivalent to choosing the density profile in the form

$$2^{(n+1)}\pi\rho(r) = \frac{b[(n+1)+ar^n]}{(1+ar^n)^2}.$$

(30)

Equation (14) for the above matter distribution takes the form

$$\frac{n(n-1)}{2r^2}\left[1 - \frac{\psi^2}{C_3^2}\right] - \frac{n\psi\psi'}{C_3^2 r} = \frac{b[(n+1)+ar^n]}{2^{(n-2)}(1+ar^n)^2},$$

(31)

whose solution is given by

$$\psi = \sqrt{C_3^2 - \frac{br^2C_3^2}{n2^{(n-3)}(1+ar^n)} + \frac{C}{r^{(n-1)}}}.$$

(32)

Thus the metric functions are obtained as

$$e^\vartheta = C_2 r^2,$$

(33)

$$e^\lambda = C_3^2\left[C_3^2 - \frac{br^2C_3^2}{n2^{(n-3)}(1+ar^n)} + \frac{C}{r^{(n-1)}}\right]^{-1}.$$

(34)

Note that the metric functions are regular at the centre if we set $C = \mathbf{0}$.
The radial and tangential pressures are obtained as

$$8\pi p_r = -\frac{n(n-1)}{2r^2} + \frac{n(n+1)}{2r^2 C_3^2}\left[C_3^2 - \frac{2br^2C_3^2}{n2^{(n-3)}(1+ar^n)} + \frac{C}{r^{(n-1)}}\right],$$

(35)

$$8\pi p_t = \frac{n(n-1)}{2br^2C_3^2}\left[C_3^2 - \frac{br^2C_3^2}{n2^{(n-3)}(1+ar^n)} + \frac{C}{r^{(n-1)}}\right] + \frac{n(1-n)C}{2r^{(n+1)}C_3^2} + \frac{b[(n-2)ar^n-2]}{2^{(n-2)}(1+ar^n)^2} - \frac{(n-1)(n-2)}{2r^2},$$

(36)

and the measure of anisotropy is given by

$$8\pi\Delta = \frac{n-1}{r^2} + \frac{n}{r^2C_3^2}\left[C_3^2 - \frac{br^2C_3^2}{n2^{(n-3)}(1+ar^n)} + \frac{C}{r^{(n-1)}}\right] + \frac{n(1-n)C}{2r^{(n+1)}C_3^2} + \frac{b[(n-2)ar^n-2]}{(1+ar^n)^2}.$$

(37)

At the boundary, radial pressure vanishes (p_r (r = R) = 0), which gives

$$\frac{n(n+1)}{2R^2C_3^2}\left[C_3^2 - \frac{bR^2C_3^2}{n2^{(n-3)}(1+aR^n)} + \frac{C}{R^{(n-1)}}\right] - \frac{n(n-1)}{R^2} = \mathbf{0}.$$

(38)

Equation (38) determines the radius R of the star.
We, thus, obtain all the physical parameters in simple analytic forms. Through energy density is regular throughout the interior of a star, the two pressures still remain singular in this model at r = 0 in the framework of higher dimension space time. The characteristic of the model are shown in the following figures (1) to (7) in the framework of five dimensional space time (i.e., n=3).
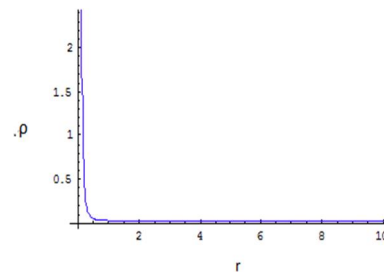

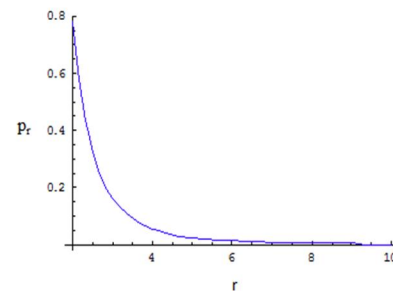Figure 1: The density parameter ρ is shown against r for a = 0.1, C = 0.5, C_3 =0.1, b=0.03

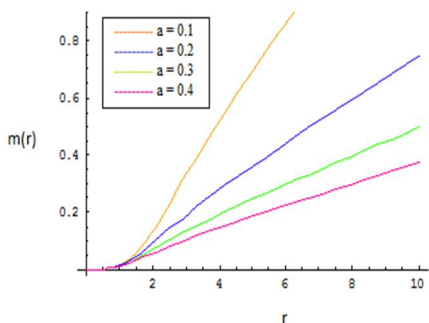Figure 2: The radial pressure p_r is shown against r for a = 0.1, C = 0.5 , C_3 =0.1, b=0.03

Figure 3: The mass parameter m(r) is shown against r for a =0.1, 0.2, 0.3, 0.4, C = 0.5, C_3 = 0.1, b = 0.01
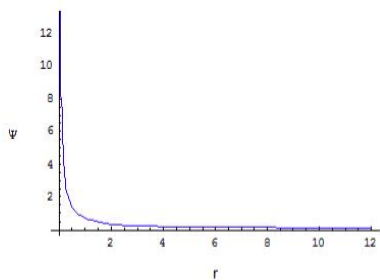

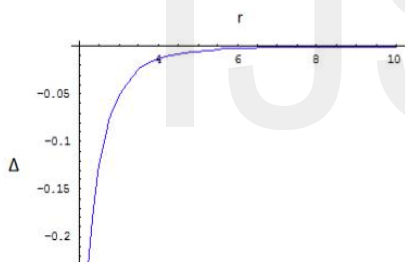Figure 4: The conformal parameter ψ(r) is shown against r for a = 0.1, C =0.5, C_3 = 0.1, b = 0.03

Figure 5: The anisotropy Δ = p_t − p_r is shown against r for a = 0.1, C =0.5, C_3 = 0.1, b = 0.03
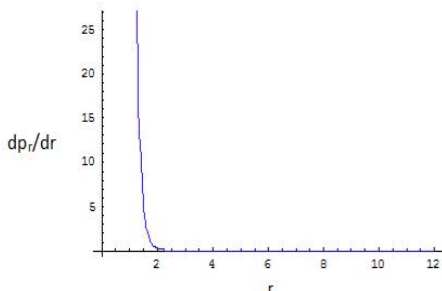

Figure 6: The radial pressure gradient dp_r /dr is shown against r for a = 0.1,C = 0.5,C_3 = 0.1,b = 0.03
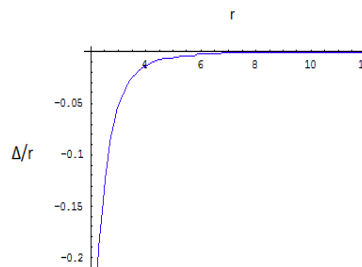

Figure 7: The force parameter Δ/r is shown against r for a = 0.1, C = 0.5, C_3 = 0.1, b = 0.03

3. Conclusion

This work has generalized to higher dimension the well known results in four dimensional space time. It is found that they may be significant difference in principle at least to analogous situation in four dimensional space time. We have provided new way of solutions for anisotropic star admitting conformal motions. These are obtained by taking the energy density ρ and mass verification profiles in two different cases in the framework of higher dimensional space time. We have total pressure of the fluid is decomposed into two pressures terms, the radial pressure p_r and transverse pressure p_t . The solutions obtained here are in simple form and can be used to study the physical behavior of compact anisotropic star. For physically meaningful solutions, it is imperative to study the behavior of physical parameters like energy density, mass, force and gradient pressure inside the star. In this model all these parameters are well behaved as shown in figures (1) to (7) in the framework of higher dimensional space-time.

However due to the effect of higher dimension the solutions suffer from a central singularity problem i.e., r = 0. This work is the generalization of the work obtained earlier by Rahaman et al. (2010a) in four-dimensional space-time.

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